ON THE THREE-CUSPED HYPOCYCLOIDS FULFILLING CERTAIN ASSIGNED CONDITIONS

BY

OTTO JOSEPH RAMLER, M.A.

A DISSERTATION

SUBMITTED TO THE CATHOLIC UNIVERSITY OF AMERICA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

STANFORD UNIVERSITY LIBRARIES
WITHDRAWN

WASHINGTON, D. C.
JUNE, 1918

3781
C36R
UNCLASSED

Digitized by the Internet Archive
in 2025

https://archive.org/details/onthreecuspedhyp0000otto

ON THE THREE-CUSPED HYPOCYCLOIDS FULFILLING CERTAIN ASSIGNED CONDITIONS

BY

OTTO JOSEPH RAMLER, M.A.

A DISSERTATION

SUBMITTED TO THE CATHOLIC UNIVERSITY OF AMERICA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

WASHINGTON, D. C.
JUNE, 1918

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

INTRODUCTION.

Within the past century Steiner* and Cremona† have given to the mathematical world much information on the properties of the three-cusped hypocycloid. In 1857 Steiner stated without proof a number of remarkable properties of this curve and eight years later Cremona showed how these properties may be proved by the application of the methods of Cayley and Hesse to the general theory of plane point and line cubics. The curve has been studied more recently by Professor Morley‡ by whom it was named the deltoid. A year later Dr. Converse§ showed that the locus of centers of a system of deltoids inscribed to a triangle is a straight line perpendicular to the Euler line of the triangle at the center of the Feuerbach, or nine-point circle of the triangle. The locus of cusps of the same system of curves is also shown to be a point-cubic with conjugate point. The question naturally arises now, what are the loci of centers and cusps of deltoids satisfying any three conditions, and imposing four conditions, how many deltoids can be found to satisfy those four conditions?||

The problems which have been considered in this paper are:

I. Given three conditions, find specified loci, as

1. Given three lines; find the locus of centers and cusps.

* J. Steiner: Über eine besondere Curve dritter Klasse (und vierten Grades), *Crelle*, Vol. 53, p. 231 (1857).

† M. L. Cremona, à Bologne: Sur l'hypocycloide à trois rebroussements, *Crelle*, Vol. 64, p. 101 (1865).

‡ F. Morley: Orthocentric properties of the plane N-line, *Transactions Amer. Math. Soc.*, Vol. 4, p. 1 (1903).

§ H. A. Converse: On a system of hypocycloids of Class three inscribed to a given three-line and some curves connected with it. *Annals of Math.*, Second Series, Vol. 5, No. 3, pp. 105–139 (1904).

|| The deltoid is a rational quartic curve. Its three cusps count for two conditions each; since it has three cusps, by Plücker's equations it must have a double tangent. The line at infinity is this double tangent, thus accounting for two more conditions; the curve also goes through the circular points, I and J. The number of conditions therefore that the deltoid can stand is $14-3\times 2-2-2=4$ conditions.

2. Given the center and a line; find the locus of cusps.
3. Given the center and a point; find the locus of cusps.
4. Given a cusp and a line; find the locus of centers and the other two cusps.
5. Given a cusp and a point; find the locus of centers.
6. Given two lines and a point; find the locus of centers.

II. Given four conditions, find how many solutions there are as

a. Given four lines, how many deltoids are there? How many real?
b. Given the center and two lines; apply the same questions.
c. Given a cusp and two points.
d. Given a line, a point, and the center.
e. Given the center and two points.
f. Given a cusp and two lines.
g. Given three lines and a point.
h. Given a cusp, a line, and a point.

THE COÖRDINATE SYSTEM. FUNDAMENTAL EQUATIONS.

The coördinate system which lends itself most readily to such problems is the system of conjugate, or circular coördinates, in which a point is designated as the extremity of a vector drawn from a fixed point, called the origin. With this vector is also associated its conjugate, *i. e.*, its reflection in the axis of reals. If we let x and y represent the vector and its conjugate respectively, then the system of conjugate coördinates is obtained from the rectangular system of Cartesian coördinates by the following transformation,

$$x = X + Yi, \qquad y = X - Yi,$$

where (X, Y) are the rectangular coördinates of the point, and i denotes as usual $\sqrt{-1}$. Regarding the deltoid as generated by a circle rolling inside another circle three times as large, the map-equation of the curve is

$$x = x_0 + r\,(2t + 1/t^2),$$

where x_0 is the center, and r is the orientation. This map-equation involves its conjugate,

$$y = y_0 + s\left(\frac{2}{t} + t^2\right),$$

where s is the conjugate of r.

These equations may be considered as the parametric equations of the curve. The elimination of the parameter t from the two equations gives the conjugate equation of the curve,

$$4\left(\frac{x - x_0}{r}\right)^3 + 4\left(\frac{y - y_0}{s}\right)^3 - \left(\frac{x - x_0}{r}\right)^2\left(\frac{y - y_0}{s}\right)^2 - 18\left(\frac{x - x_0}{r}\right)\left(\frac{y - y_0}{s}\right) + 27 = 0.$$

The equation of a tangent line at the point t, is

$$x = \frac{r}{ts}y + x_0 - \frac{ry_0}{ts} + rt - \frac{r}{t^2}.$$

PROBLEM I. Given three lines to find the locus of centers of deltoids tangent to them. How many deltoids can touch four lines?

H. A. Converse has shown that the locus of centers of deltoids touching three lines is a straight line. The locus of centers of deltoids touching two of these lines and a different third line, will be another straight line. These center-loci intersect in but one point which is the center of the only deltoid that can be drawn tangent to four lines. The number of deltoids that can be inscribed to a definite four-line is therefore one, and it is necessarily real.

PROBLEM II. Given the center and a line, to find the locus of cusps.

Let the origin be the center O, Fig. 1, and the line MN have as its equation $x + y = 2$. Identifying the equation $x + y = 2$ with the equation of the tangent $t^2sx - ryt - t^3rs + rs = 0$, we obtain the relation,

$$r^3 + s^3 + 2rs = 0. \tag{1}$$

The map-equation of a deltoid is

$$x = r\left(2t + \frac{1}{t^2}\right) + x_0;$$

Cusps are given by $dx/dt = 0$, that is for points whose parameters are $t = 1, \omega, \omega^2$, where ω and ω^2 are the two imaginary cube roots of unity. Choose the cusp given by $t = 1$, *i. e.*, $x = 3r$. Substituting in (1) we have the cusp locus

$$x^3 + y^3 + 6xy = 0. \tag{2}$$

Equation (2) can be written

$$(x + y - 2)(x + \omega y - 2\omega^2)(x + \omega^2 y - 2\omega) + 8 = 0. \tag{3}$$

Hence

$$x + y = 2,$$
$$x + \omega y = 2\omega^2,$$
$$x + \omega^2 y = 2\omega$$

are asymptotes; the line at infinity is the line of flexes; the origin is an isolated double point.

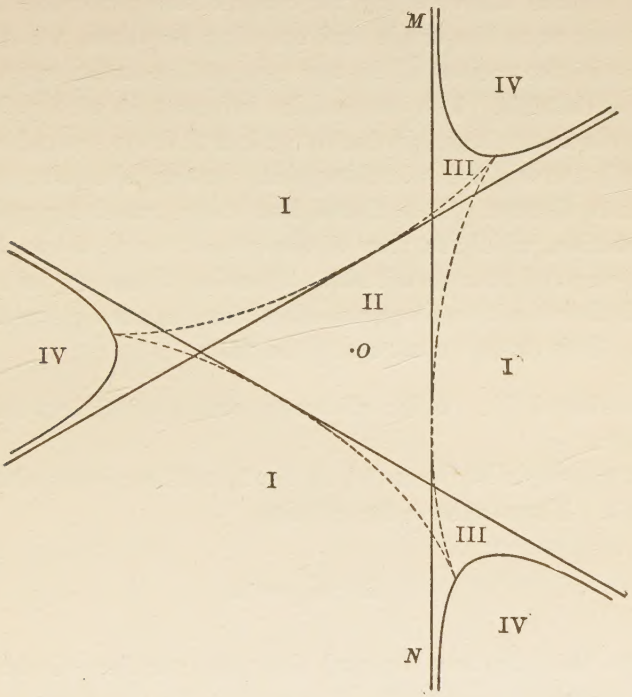

FIG. 1.

If the given line were $x = t_1 y + a$, the cusp locus would be

$$(x - t_1 y - a)\,(x - \omega t_1 y - a\omega^2)\,(x - \omega^2 t_1 y - a\omega) + a^3 = 0. \quad (4)$$

This cubic also has the origin as an isolated double point with the circular rays as tangents thereat. We are now able to answer question II (*b*).

PROBLEM II (*b*). Given the center and two lines, how many deltoids?

Let the origin be the given point, and the two lines have equations $x + y = 2$, and $x = ty + a$, respectively. By problem (2) the cusp-loci are similar cubics. It should be stated here that these cubics consist of three branches, each branch carrying a cusp of the deltoid; this should be expected due to the circular symmetry of the deltoid. It is important to bear this fact in mind because when seeking the cusps of the particular deltoids with centers at the origin and touching two lines, we take the common intersections of the two cusp cubics as the cusps of the desired deltoids. The cusp-cubics intersect in nine points; of these, six are at the origin due to the fact that the two cubics have not only the origin as a double-point, but they also have common tangents thereat. This allows but three remaining points of intersection, which, in view of the remark made above, are the three cusps of but one deltoid. Therefore there can be but one deltoid drawn with a given center and touching two given straight lines. This deltoid must always be real.

PROBLEM I (3). Given the center and a point; find the locus of cusps.

Let the origin be the center *O*, Fig. 2, and the given point *P* the point 1. Then we have the relation

$$2rt + \frac{r}{t^2} = 1. \tag{1}$$

Choose the cusp $x = 3r$, and the map-equation of the cusp-locus becomes

$$x = \frac{3t^2}{2t^3 + 1}. \tag{2}$$

This is a sextic with cusps at $1, \omega, \omega^2$.

The conjugate equation of it is

$$4(x^3 + y^3) - 6x^2y^2 - 3xy + x^3y^3 = 0. \tag{3}$$

This curve is the inverse of the standard deltoid with respect

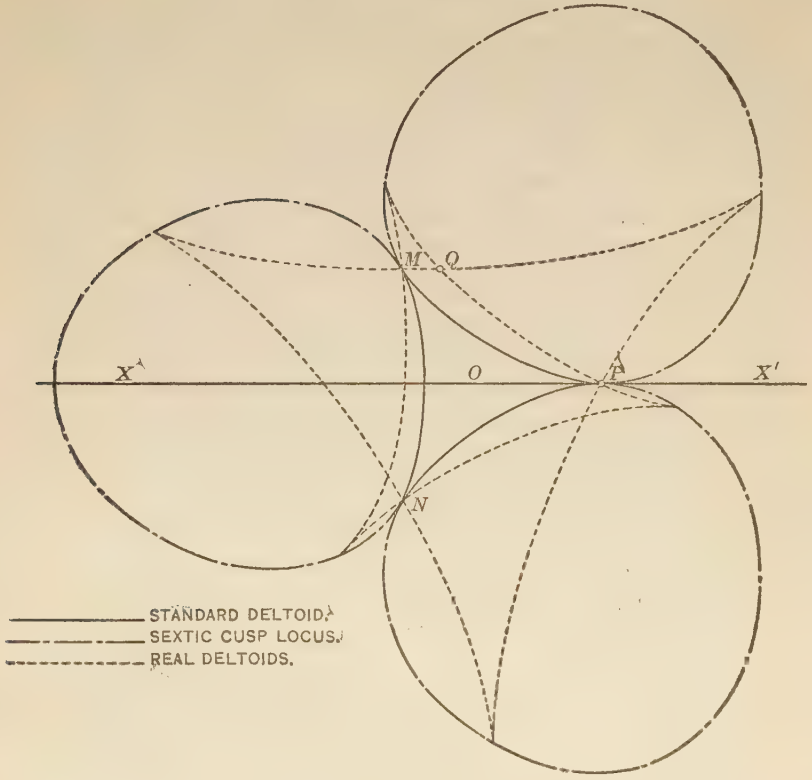

FIG. 2.

to the unit circle $xy = 1$, as may be seen by applying the transformation $x = 1/y$ to the equation of the standard deltoid

$$4(x^3 + y^3) - 3x^2y^2 - 6xy + 1 = 0. \qquad (4)$$

The sextic (3) has cusps at $1, \omega, \omega^2$, triple points at I and J, and conjugate point at the origin.

If the given point be taken to be p instead of 1, the sextic is

$$4(qx^3 + py^3) - 6pqx^2y^2 - 3p^2q^2xy + x^3y^3 = 0, \cdots \qquad (5)$$

where q is the conjugate of p.

The sextic (5) has triple points I and J, with $4q^3 + y^3 = 0$, and $4p^3 + x^3 = 0$ respectively as tangents thereat. These data yield the solution to problem II (e).

PROBLEM II (*e*). Given the center and two points, how many deltoids can be drawn having the given center and passing through the two given points? Under what circumstances will the deltoids all be real?

Taking the origin as center, and the points 1 and p as the given points and taking the results of problem I (3), the cusp-sextics will intersect in thirty-six points. Among these the common triple points at I and J count for nine each; the curves have the same double point at the origin with the same imaginary tangents, thus accounting for six more intersections. There remain therefore twelve other intersections. Remembering, as before, that among these points there are sets of three points each, each set representing a unique deltoid. There can therefore exist but four deltoids passing through two given points with centers fixed.

We shall now discuss the reality of the four deltoids satisfying the above conditions.

If the deltoid passes through 1 and has center 0, we have

$$4(r^3 + s^3) - rs - 18r^2s^2 + 27r^3s^3 = 0. \tag{1}$$

If t is the parameter giving the point p on the cusp-sextic then

$$r = \frac{pt^2}{2t^3 + 1}, \qquad s = \frac{qt}{2 + t^3}. \tag{2}$$

Substituting, and calling t^3 simply t, we get

$$\begin{aligned} t^4(4p^3 - 4pq) &+ t^3(24p^3 + 32q^3 - 20pq - 36p^2q^2) \\ &+ t^2(48p^3 + 48q^3 - 33pq - 90p^2q^2 + 27p^3q^3) \\ &+ t(32p^3 + 24q^3 - 20pq - 36p^2q^2) + (4q^3 - 4pq) = 0. \end{aligned}$$

Now transform the unit circle into the axis of reals by the transformation

$$t = -\frac{x - i}{x + i}.$$

This gives

$$\begin{aligned} x^4(4p^3 + 4q^3 + pq + 18p^2q^2 - 27p^3q^3) &+ 4x^3i(8p^3 - 8q^3) \\ &+ 6x^2(-3)(4p^3 + 4q^3 - pq - 10p^2q^2 + 3p^3q^3) \\ &- 27(4p^3 + 4q^3 - 3pq - 6p^2q^2 + p^3q^3) = 0. \end{aligned}$$

Let

$$4p^3 + 4q^3 - 3pq - 6p^2q^2 + p^3q^3 = A.$$

The quartic becomes, letting $S = pq$,

$$x^4(A + 4S + 24S^2 - 28S^3) + 4x^3i(8p^3 - 8q^3) + 6x^2(-3)(A + 2S - 4S^2 + 2S^3) - 27A = 0. \quad (3)$$

This shows the existence of four deltoids satisfying the assigned conditions. From the theory of the binary quartic we know that two roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$ will be equal, where the coefficients a, b, c, d, e are all real, providing the discriminant $\Delta = I^3 - 27J^2 = 0$, where

$$I = ae - 4bd + 3c^2, J = ace + 2bcd - ad^2 - b^2e - c^3.$$

The invariants for the quartic (3) are

$$I = 108S^2(S - 1)[8A + (S - 1)^3],$$

$$J = -216S^3\{[(8A + (S - 1)^3][A + 2(S - 1)^3] + 3(S - 1)^3[A - (S - 1)^3]\}.$$

It is known that if the discriminant

$\Delta < 0$, the quartic has two real and two imaginary roots;

$\Delta > 0$, the quartic has 0 or 4 real roots;

$\Delta > 0$, and besides $b^2 - ac > 0$, and $12(b^2 - ac)^2 - Ia^2 > 0$, the quartic has 4 real roots.*

The curve $\Delta = 0$ will therefore separate the plane into the regions which we seek.

$$\Delta = -432^3S^6A[A - (S - 1)^3]^3 = 0. \quad (4)$$

The locus (4) consists of the curves $A = 0$, and $A - (S - 1)^3 = 0$.

Now $A = 0$ is the cusp-sextic, and $A - (S - 1)^3$ is the standard deltoid. For points p on these two curves there are two of

* Cf. Halphen: Traité des Fonctions Elliptiques et Leurs Application, Paris, 1886, Première Partie, p. 123.

the four deltoids coincident. If p is not on these curves we have, testing for various points of the plane,

$$p = 1/2, \Delta > 0, \qquad b^2 - ac > 0, \qquad 12(b^2 - ac)^2 - Ia^2 < 0,$$

hence none are real inside the standard deltoid, PMN, Fig. 2. $p = -1$, $I = 0$, $\Delta < 0$, hence two deltoids only can be real in the region bounded by the standard deltoid and the sextic. $p = 2$, $\Delta > 0$, $b^2 - ac < 0$, hence none are real in the region outside the sextic.

See Fig. 2 for the case where the two real deltoids are drawn.

Problem II (d). Given the center, a line and a point, to find the number of deltoids and determine their reality.

Let the center be the origin 0 (Fig. 1); let $x + y = 2$ be the equation of the line MN, and p the point P. Identifying these equations with the fundamental equations we have

$$rt - \frac{r}{t^2} = 2, \qquad r^3 + s^3 + 2rs = 0, \tag{1}$$

$$p = 2rt + \frac{r}{t^2},$$

where t is the parameter giving point P.

Hence

$$r = \frac{pt^2}{2t^3 + 1}, \qquad s = \frac{qt}{2 + t^3}.$$

Substituting in (1) we get a quartic in t^3. Calling t^3 simply t, this quartic is

$$t^4(p^3 + 8pq) + t^3(6p^3 + 8q^3 + 40pq) + t^2(12p^3 + 12q^3 + 66pq) + t(8p^3 + 6q^3 + 40pq) + (q^3 + 8pq) = 0.$$

Transforming the unit circle into the axis of reals by the transformation

$$t = -\frac{x - i}{x + i},$$

and letting

$$p^3 + q^3 + 6pq = A, \qquad pq = S,$$

we get

$$x^4(A - 8S) + 4x^3i(2p^3 - 2q^3) + 6x^2(-3)(A - 4S) - 27A = 0, \quad (2)$$

The invariants for this quartic are

$$I = 3^3 \cdot 2^4 \cdot S^2,$$
$$J = 3^3 \cdot 2^4 \cdot S^3 (A - 4),$$
$$\Delta = -3^9 \cdot 2^8 \cdot S^6 \cdot A(A - 8).$$

Here again, as before, $\Delta = 0$ divides the plane into the regions sought. Tests for points other than those on the curves $A = 0$, and $A = 8$, give the following results:

Point 2, $\Delta < 0$, hence there are two real and two imaginary deltoids; this region is indicated in Fig. 1 by I.

Point $1/2$, $\Delta > 0$, $b^2 - ac = -9/16 < 0$; $12(b^2 - ac)^2 - Ia^2 < 0$; hence if point p lies in the region indicated in the figure by II, no real deltoids can be drawn to satisfy the stated conditions.

Point $-5/2$, $\Delta > 0$, $b^2 - ac > 0$, $12\ (b^2 - ac)^2 - Ia^2 > 0$; therefore 4 real deltoids in the region marked III.

Point -4, $\Delta < 0$, hence but two are real in the region IV.

PROBLEM I (4). Given the cusp and a line to find the locus of centers and other cusps.

Let the point 3 be the fixed cusp P, and the line have the equation $x + y = 2$. The map equation of the deltoid with center at x_0 and orientation r, is

$$x = x_0 + 2rt + \frac{r}{t^2}. \quad (1)$$

The tangent at the point whose parameter is t is

$$t^2sx - ryt - t^2sx_0 + ry_0t - t^3rs + rs = 0. \quad (2)$$

Take the cusp given by $t = 1$, in (1), namely

$$x_c = x_0 + 3r = 3; \qquad i.e., \qquad r = \frac{3 - x_0}{3}, \qquad s = \frac{3 - y_0}{3}.$$

Identifying the tangent with the given line, we get after

simplifying and dropping subscripts, the equation of the center locus as

$$(3-x)^3+(3-y)^3+(x+y-2)^3-3(3-x)(3-y)(x+y-2)=(x+y-2)^3. \quad (3)$$

Locus (3) is a cubic with the given line as the line of flexes, and the given point 3 as a conjugate point.

Similarly it can be shown that the loci of the two moving cusps are similar cubics, each having the given line as a flex tangent and point 3 as a conjugate point.

Problem II (f). Given a cusp and two lines, to find the number of deltoids.

If the cusp is the point 3, and the given line is $x = ty$, the locus of centers is a cubic with the given line as line of flexes and the point 3 as conjugate point with the circular rays as imaginary tangents thereat. In problem I (4) it was shown that if the given cusp is 3 and the line has equation $x + y = 2$, the center locus is a cubic having the point 3 as a conjugate point with the circular rays as tangents. These two cubics intersect in only one real point, as may be easily shown. Two such cubics may be represented in Cartesian rectangular coördinates by the equations

$$3y^2 = -x^2(2x+3),\cdots \quad (1)$$

$$c(x^2+y^2) = (x\sin\theta + y\cos\theta)^3,\cdots \quad (2)$$

where θ is the angle that the line of flexes in the second cubic makes with the X-axis.

Solving equations (1) and (2) simultaneously we get the binary cubic

$$\frac{2cx^3}{3}+(x\sin\theta+y\cos\theta)^3=0. \quad (3)$$

The discriminant of (3) is

$$\Delta = 4c^2\cos^6\theta\left(\sin^3\theta+\frac{2c}{3}\right)^2>0$$

for all values of c and θ. Hence but one real deltoid can be drawn having its cusp at a given point and at the same time tangent to two fixed lines.

PROBLEM I (5). Given a cusp and a point, to find the locus of centers.

Let the cusp be at the origin, and the given point at -1, then we have

$$-1 = x + 2rt + \frac{r}{t^2},$$

where t is the parameter giving the point -1 and x is the center. Since the cusp is at the point 0, define it as $0 = x + 3r$, which gives $r = -x/3$.

Substituting above we get

$$\frac{3(1+x)}{x} = 2t + \frac{1}{t^2}$$

and its conjugate

$$\frac{3(1+y)}{y} = \frac{2}{t} + t^2.$$

Eliminating t from these two equations, the desired center-locus assumes as its equation

$$9xy(x-y)^2 + 4(x^3+y^3) - 6xy(x+y) - 3xy = 0. \quad (1)$$

The origin is a conjugate point with the circular rays as imaginary tangents. The curve is a circular quartic, tangent to the line at infinity where the axis of reals cuts it.

If we had taken the given point in a more general position say at $-p$ and the cusp at the origin, equation (1) would have been modified so as to become

$$9xy(qx-py)^2 + 4(q^3x^3 + p^3y^3) - 6xypq(qx+py) - 3p^2q^2xy = 0. \quad (2)$$

This curve is of the same type as (1). They are self-dual, being of order and class 4, with two cusps and two flexes, an isolated double point and double tangent.

PROBLEM II (h). Given a cusp, a line and a point, to determine the number of deltoids. Discuss the reality.

Let the origin be taken as cusp, the point -1 as the given point, and the line whose equation is $bx + ay = ab$ as the given line, where a is the vector to the reflection of the origin in the

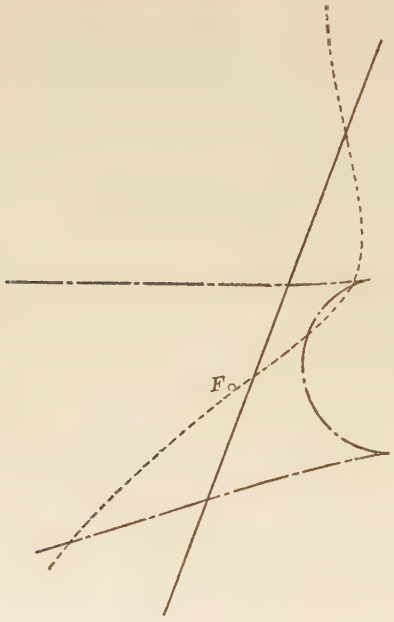

FIG. 3.

line and b its conjugate. Then by problem I (4) the center locus, given cusp and line, is the cubic with the given line as line of flexes, and the fixed cusp as conjugate point; its equation is

$$(bx + ay)^3 = 3a^2b^2xy. \tag{1}$$

The center locus for the point-cusp case by problem I (5) is the circular quartic

$$9xy(x - y)^2 + 4(x^3 + y^3) - 6xy(x + y) - 3xy = 0. \tag{2}$$

Solving (1) and (2) we should get the centers of the deltoids satisfying the four conditions of our problem. The solution of (1) and (2) gives 12 points of intersection, but the common

double point and imaginary tangents at the origin reduce this number to six. An inspection of Fig. 3 will render it fairly evident that at most but four intersections of the two curves can be real. Fig. 3 is drawn for the case where the four intersections are real.

An interesting case which will throw some light on a further study of the reality of the four deltoids is the case when the given line goes through the fixed point -1. For, consider the pencil of lines through -1, namely

$$bx + ay = ab,$$

where

$$b = \frac{-a}{1+a}.$$

Substituting in (1) we have,

$$[(1+a)y - x]^3 = 3a(1+a)xy. \qquad (4)$$

This is the equation of the cubic center locus where the given line is a line in this pencil. By differentiating (4) with respect to a, and eliminating a from (4) and its differentiated form, we get the envelope of the system of center-cubics to be

$$9xy(x-y)^2 + 4(x^3+y^3) - 6xy(x+y) - 3xy = 0,$$

which is the circular quartic (2), or the center locus of the point-cusp problem. Hence, if the given point and given line coincide, the two center-loci are tangent to each other, and therefore two of the four centers satisfying the given conditions will coincide, that is there will be among the four real solutions two coincident solutions. This is illustrated in Fig. 4, the center loci touching at point T.

If the given line is perpendicular to the line joining the fixed points F and P, Fig. 4, it can be easily shown that if it is perpendicular to PF (the axis of reals) at any point between the points -1, and $-1/2$, the cubic will cut the quartic in four real points. If the line is perpendicular at $-1/2$, the cubic cuts the quartic at its two cusps c and c' (Fig. 4), thus giving two pairs of coincident solutions. If the line is perpendicular at any point to the right of $-1/2$, there are no real solutions; if it is perpen-

dicular at a point to the left of -1, there are always two and only two real solutions.

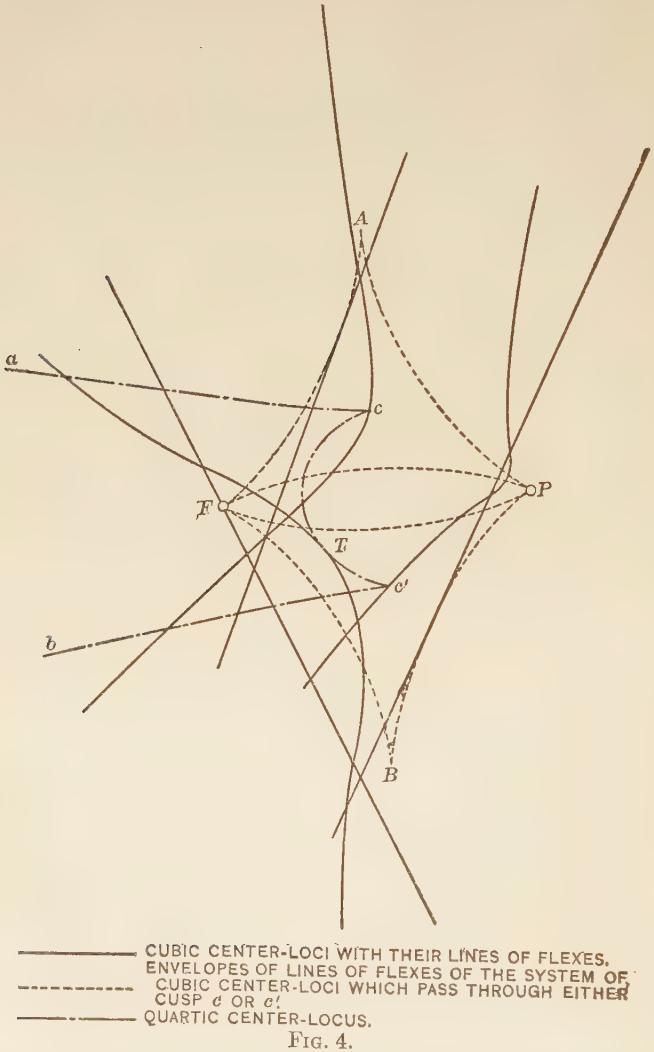

FIG. 4.

Let us carry the discussion a bit further. Take the line of flexes (the given line) to be $hx + ky + 1 = 0$, where (h, k) will then be the line coördinates of the given line. The equation of the cubic center locus then will be

$$(hx + ky)^3 + 3hkxy = 0. \tag{5}$$

If we insist that this cubic shall go through the cusp c (Fig. 4), whose point coördinates are

$$x = \frac{1}{\omega^2 - 1}, \qquad y = \frac{1}{\omega - 1},$$

we get as a result

$$(k - \omega^2 h)^3 + \omega(\omega - 1)hk = 0. \tag{6}$$

which is the line equation of a deltoid with center at c, and cusps at F and P.

The interpretation of this result is as follows: If the given line be tangent to this deltoid AFP (Fig. 4) and the fixed points be taken as before to be F and P the cubic-center locus will pass through the cusp c, thus giving again a pair of coincident solutions among the four real deltoids. A similar result is obtained for cusp c'. The two deltoids are shown in Fig. 4 as FAP and FBP. If the given line be tangent to either arc AF or FB the corresponding center loci will cut the quartic $acc'b$ in two real distinct points other than c, whereas if the given line is tangent to arc AP or to PB the only real intersection of the cubic center loci and the quartic is c or c' respectively.

Problem II (c). Given the cusp and two points, how many deltoids can be drawn?

Let the origin be the cusp and points 1 and p the given points, then using the information of problem I (5) we have as the center locus for cusp and point 1 the circular quartic (1) of that problem; for point p the center locus is the circular quartic (2) of the same problem. These two curves intersect in 16 points of which the common double point and imaginary tangents at the origin account for six and the points I and J reduce the number by two more. It appears therefore that the number of deltoids that can be drawn having a common cusp and going through two fixed points is eight. It will be seen that at most but four of this number can be real.

The equation of deltoid AFP (Fig. 4) is

$$4\left[\frac{x^3}{3\omega^2(\omega^2-1)}+\frac{y^3}{3\omega(\omega-1)}\right]+3\left(\frac{x}{\omega-1}-\frac{y}{\omega^2-1}\right)^2 - 2xy\left(\frac{x}{\omega-1}+\frac{y}{\omega^2-1}\right)-x^2y^2=0,$$

which is the same as would be obtained from equation (2) problem I (5) by making the substitutions

$$p=x,\ q=y,\ x=\frac{1}{\omega^2-1},\qquad y=\frac{1}{\omega-1}.$$

The point

$$x=\frac{1}{\omega^2-1},\qquad y=\frac{1}{\omega-1},$$

is the center of the deltoid AFP (Fig. 4), and (p, q) is the given point of problem I (5) taken in a general position. Hence the following conclusion: If a second point Q be given besides F, P being the fixed cusp, and Q lies on the deltoid AFP, then the center locus for Q with P as fixed cusp will always pass through the cusp c of the center locus $acc'b$. Therefore one curve defining the regions of reality is the deltoid AFP. Similarly another curve is the deltoid FBP.

To determine other curves defining the regions reality, consider the problem to find the cusp locus, given a fixed cusp P and a given point F.

A cusp x_c is given by

$$x_c=x_0+3r.$$

Let this be the fixed cusp P at the origin, x_0 being the center. Therefore

$$x_0+3r=0,\qquad i.\ e.,\qquad r=-\frac{x_0}{3}.$$

A second cusp is given by

$$x_k=x_0+3\omega^2 r.$$

Therefore

$$x_0 = \frac{x_k}{1-\omega^2} = \frac{x_k(1-\omega)}{3}.$$

FIG. 6.

Since the center x_0 has for its locus the circular quartic

$$9xy(x-y)^2 + 4(x^3+y^3) - 6xy(x+y) - 3xy = 0,$$

we shall obtain the desired cusp-locus by making in this equation the substitution

$$x = \frac{x_k}{1 - \omega^2}, \qquad y = \frac{y_k}{1 - \omega},$$

and dropping the subscript k.

The new curve is also a circular quartic, in fact the same type as the center locus since each vector is simply divided by $1 - \omega^2$ or by $1 - \omega$ as the case may be. Likewise the cusp locus for the third cusp is the same circular quartic turned through $+ 60°$. The regions of reality are now entirely determined. An inspection of Fig. 6 will indicate the complete solution of the problem. Conceive a deltoid PDD' (Fig. 6) to move so that its cusp presently will assume the positions PCC', PMF, PBB', PAA'.

During this process it will have been noticed that the side PD when sweeping over the region $DMRFV$ would pass but once through a point Q if Q were in the region $DMRFV$, whereas if Q were in the region $FRMF$, the side PD would pass through it twice.

Similarly for the side DD'. Hence if point Q lies in $MRFM$, four deltoids can be drawn through it satisfying the four conditions, whereas but two deltoids can be drawn if Q should lie in the region $DMRFV$. Similar reasoning can show that if Q lies in the region VFD, $FSNF$, $FLPKF$, four deltoids can be drawn; whereas if Q lies in the regions $DFSNW$, $MFLPM$, $NFKPN$, but two deltoids can be drawn. For other regions no deltoid can be drawn.

Problem I (6). Given two lines and a point, to find the locus of centers.

Let the two lines have equations $x = ty$ and $x = y/t$, and the vector to the given point be p with q as its conjugate. Identifying the equations $x = ty$ and $x = y/t$ with the equation of a tangent line as given on page 2 we have

$$x_0 - ty_0 + \frac{A}{t} - Bt^2 = 0,$$

$$x_0 - \frac{y_0}{t} + At - \frac{B}{t^2} = 0,$$

where $A = r^2/s$, $B = s^2/r$ and (x_0,y_0) is the center.

Solving these two equations for A and B, we have

$$A = \frac{t(t^4L_2 - L_1)}{1 - t^6}$$

$$B = \frac{t^2(L_2 - t^2L_1)}{1 - t^6}, \cdots \tag{1}$$

where

$$L_1 \equiv x_0 - ty_0, \qquad L_2 \equiv x_0 - y_0/t.$$

The general equation of a deltoid with center (x_0,y_0) and orientation r, is

$$4\left(\frac{x - x_0}{r}\right)^3 + 4\left(\frac{y - y_0}{s}\right)^3 - \left(\frac{x - x_0}{r}\right)^2\left(\frac{y - y_0}{s}\right)^2 - 18\left(\frac{x - x_0}{r}\right)\left(\frac{y - y_0}{s}\right) + 27 = 0.$$

If in this equation we multiply through by r^2s^2, the equation can be written

$$4(p - x_0)^3B + 4(q - y_0)^3A - (p - x_0)^2\,(q - y_0)^2 - 18AB(p - x_0)\,(q - y_0) + 27A^2B^2 = 0.$$

Substituting in this equation the values of A and B given by (1), we obtain a quartic equation in (x_0, y_0) which is the equation of the center locus. This result furnishes the data required for the solution of problem II (g).

Problem II (g). Given three lines and a point, to determine the number of deltoids.

Let the three lines be the sides of the triangle ABC (Fig. 5). From the paper of Dr. Converse referred to on page 1, we learn that the locus of centers of deltoids touching the three lines is a line perpendicular to the Euler line of the triangle at a point

midway between the circumcenter and the orthocenter. The cusp locus is also shown to be a cubic with isolated double point having the sides of the triangle as flex tangents. From problem I (6) we learn that the center locus for the two-line and point problem is a quartic curve. From this it follows that there are at most four deltoids satisfying the conditions of our problem.

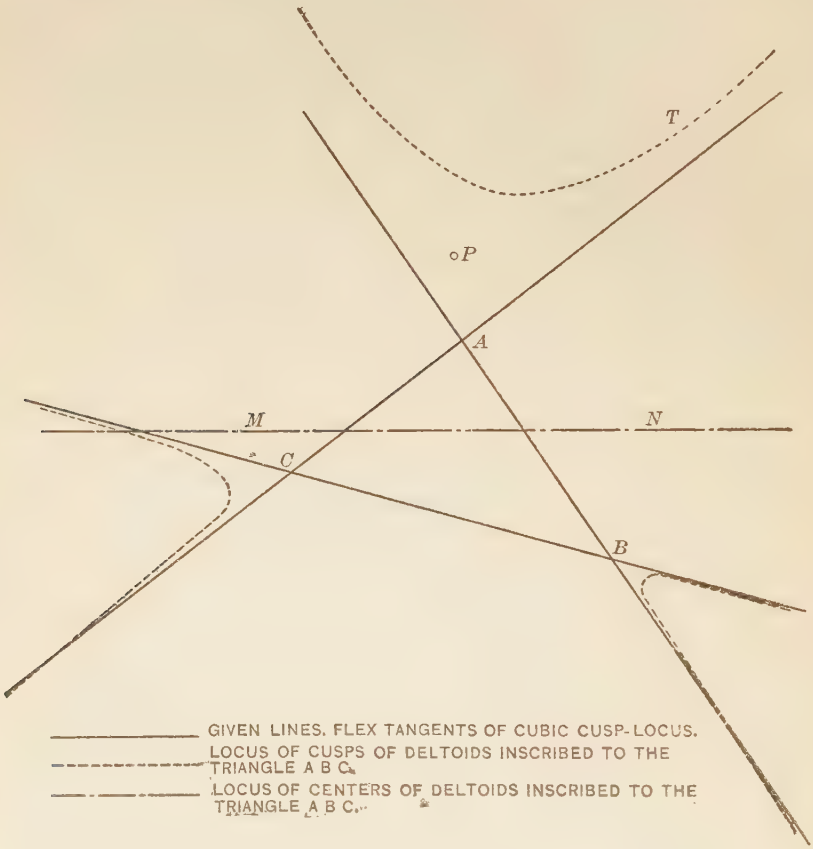

FIG. 5.

The question of reality can most readily be answered by using a theorem stated and proved by Dr. Converse in the paper mentioned above. (Theorems X and XI, pp. 128–129.) "If tangents be drawn in a given direction to the deltoids inscribed

to a triangle, the locus of the point of tangency, is a line." The envelope of this line for all possible directions is also proved to be the cubic cusp locus of our problem.

Choosing the point P in such a manner, therefore, as to be able to draw from it a tangent to the cubic, we can then consider the tangent to be a line of the cubic regarded as the envelope referred to in the theorem stated above. Let PT (Fig. 5) be one position of the line on the cubic from point P. Then PT is the locus of points of tangency of a system of parallel lines touching the system of deltoids inscribed to the triangle. Therefore one and only one deltoid of the system passes through P, since one and only one line of the system of parallel lines passes through P. There is therefore a one-to-one correspondence between the deltoids through P inscribed to the three line, and the tangents that can be drawn to the cubic cusp locus from P.

Hence the regions of the plane in which a point must lie in order that from it, four, two, or no real tangents to the cubic can be drawn are the same as the regions in which point P must lie, in order that through it, four, two, or no real deltoids can be drawn touching the three given lines. It is evident from the figure (Fig. 5) that if P lies inside the triangle ABC, no real deltoids can be drawn satisfying the four conditions. The three sides of the triangle (flex tangents of the cubic cusp locus) and the cubic cusp locus divide the plane into the regions sought. Therefore if point P should leave the triangle by crossing one side (flex tangent), through it two real deltoids can be drawn tangent to the three given lines. This region it will be noticed is bounded by the three given lines and at least a part of the cubic cusp locus. There are three such regions. If P should leave any one of these three regions by crossing either one of the three given lines or a branch of the cusp cubic, but not so as to reënter the triangle ABC, a region of the plane is entered in which P must lie in order that through it four real deltoids can be drawn touching the three given lines. There are six such regions. If point P should leave anyone of these regions by crossing either a flex tangent or the cusp cubic a region is entered where P must lie so that through it but two real deltoids can be drawn to satisfy the conditions of our problem.

BIOGRAPHY.

Otto Joseph Ramler was born in Richmond, Ind., on June 12, 1887. He received his early education in the St. Andrew's Parochial School of that city. For seven years he attended Canisius High School and College, Buffalo, N. Y. (1903–1910), graduating with the degree A.B. The year (1910–1911) was spent as a special student in mathematics and physics at the Massachusetts Institute of Technology, Boston, Mass. The following year (1911–1912) he was an instructor in mathematics at Canisius College, where the degree M.A. was conferred upon him in June. The year (1912–1913) he was instructor in mathematics at D'Youville College, Buffalo, N. Y., and for the past five years he has been a graduate student and instructor in mathematics at the Catholic University of America.

The author desires to take this occasion to express his sincere gratitude to Dr. A. E. Landry for much valuable assistance received and suggestions offered in the preparation of this dissertation and during his entire university course.